CATOLICA LISBON
SCHOOL OF BUSINESS & ECONOMICS

UNIVERSIDADE CATÓLICA PORTUGUESA

Matemática III **2011/2012 2ºSemestre**

Grupo I

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1) Resolva as equações diferenciais:

a) $xy' + 2y - 3x + (y-x)^2 = 0$ Sugestão: faça $yz = xz + 1$

b) $f(x)dy + [f'(x)y - 5f(x)f'(x)\sqrt{y}]dx = 0$

c) $[x(y-1)^3 - y^2x]y' = y(1-y)^3$

(6.5)

2) Considere a equação diferencial $y' + f(x)y^2 + g(x)y = h(x)$.

Mostre que, sendo conhecida uma solução $y_1(x)$ da equação, a mudança de variável $y = y_1 + \frac{1}{z}$ transforma a equação dada numa equação linear.

(2.0)

3) Dada a equação diferencial $e^x \sec y - tgy + y' = 0$

a) Sabendo que a equação dada admite o factor integrante $\lambda = \frac{\cos y}{e^{ax}}$, determine o valor de $a \in \mathbb{R}$.

b) Para o valor determinado na alínea a), resolva a equação dada.

(2.5)

Grupo II

4) Resolva a equação diferencial $y''' - y'' - 5y' - 3y = \frac{2}{e^x} + senx$

(2.5)

5) Determine α e $\beta \in \mathbb{R}$ de modo que a equação $x^2 y'' + \alpha x y' + \beta y = 0$

admite as soluções $y = x^2$ e $y = x^3$.

(1.5)

6) Resolva a equação diferencial

$$x^6 y'' + (3x^5 + 4x^3)y' + 4y - \frac{1}{x^4} = 0$$ Sugestão: faça $x\sqrt{t} = 1$

(2.5)

7) Resolva a equação com diferenças

$$y_{k+3} + y_{k+2} - 2y_k = k + 2^k$$

(2.5)

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a) $\boxed{xy' + 2y - 3x + (y-x)^2 = 0 \qquad y = x + \frac{1}{z}}$

$$y' = 1 - \frac{z'}{z^2}$$

$$x\left(1 - \frac{z'}{z^2}\right) + 2\left(x + \frac{1}{z}\right) - 3x + \frac{1}{z^2} = 0$$

$$\Leftrightarrow \cancel{x} - \frac{z'}{z^2}x + \cancel{2x} + \frac{2}{z} - \cancel{3x} + \frac{1}{z^2} = 0$$

$$\Leftrightarrow -z'x + 2z = -1$$

$$\Leftrightarrow z' = \frac{2}{x}z + \frac{1}{x}$$ Equação linear

$$\lambda = e^{\int -\frac{2}{x}dx} = e^{-2\ln|x|} = \frac{1}{x^2}$$

$$\frac{z}{x^2} = \int \frac{1}{x^3}\,dx$$

$$\Leftrightarrow \frac{z}{x^2} = -\frac{1}{2x^2} + c$$

$$\Leftrightarrow z = -\frac{1}{2} + cx^2$$

$$\Leftrightarrow \boxed{\frac{1}{y-x} = cx^2 - \frac{1}{2}}$$

1

b) $$\boxed{f(x)\, y' + f'(x)\, y - 5\, f(x)\, f'(x)\, \sqrt{y} = 0}$$

$$\Leftrightarrow \quad y' + \frac{f'(x)}{f(x)}\, y = 5\, f'(x) \cdot y^{1/2}$$ Eq. de Bernoulli

$$\Leftrightarrow \quad y^{-1/2}\, y' + \frac{f'(x)}{f(x)}\, y^{1/2} = 5\, f'(x)$$

$z = y^{\frac{1}{2}}$

$z' = \frac{1}{2}\, y^{-1/2}\, y'$

$2z' = y^{-1/2}\, y'$

$$2z' + \frac{f'(x)}{f(x)}\, z = 5\, f'(x)$$

$$\Leftrightarrow \quad z' + \frac{f'(x)}{2\, f(x)}\, z = \frac{5}{2}\, f'(x)$$ Eq. Linear

$$\lambda = e^{\frac{1}{2}\int \frac{f'(x)}{f(x)}\, dx} = e^{\frac{1}{2}\ln |f(x)|} = \sqrt{f(x)}$$

$$z \cdot \sqrt{f(x)} = \int \frac{5}{2} \cdot f'(x) \cdot [f(x)]^{1/2}\, dx$$

$$\Leftrightarrow \quad z \cdot \sqrt{f(x)} = \frac{5}{2}\, \frac{[f(x)]^{3/2}}{3/2} + C$$

$$\Leftrightarrow \quad z\, \sqrt{f(x)} = \frac{5}{3}\, \sqrt{f^3(x)} + C$$

$$\Leftrightarrow \quad \boxed{\sqrt{y} \cdot \sqrt{f(x)} = \frac{5}{3}\, \sqrt{f^3(x)} + C}$$

4

$$\boxed{c)\ \left[x(y-1)^3 - y^2 x\right]\frac{dy}{dx} - y(1-y)^3 = 0}$$

$$\Leftrightarrow\quad x\left[(y-1)^3 - y^2\right]dy - y(1-y)^3\,dx = 0$$

$$\Leftrightarrow\left[\frac{(y-1)^3 - y^2}{y(1-y)^3}\right]dy - \frac{1}{x}\,dx = 0$$

$$\Leftrightarrow\left[-\frac{1}{y} - \frac{y}{(1-y)^3}\right]dy - \frac{1}{x}\,dx = 0$$

$$\int\left[\frac{1}{y} + y(1-y)^{-3}\right]dy + \int\frac{1}{x}\,dx = C$$

$$\ln|y| + \frac{2y-1}{2(1-y)^2} + \ln|x| = C$$

$$\Leftrightarrow\quad \boxed{\ln|x\cdot y| + \frac{2y-1}{2(1-y)^2} = C}$$

C.A : $P\ \underbrace{y}_{u}\,\underbrace{(1-y)^{-3}}_{v'} = \frac{1}{2}y(1-y)^{-2} - P\,\frac{1}{2}(1-y)^{-2} =$

$$= \frac{y}{2(1-y)^2} - \frac{1}{2}(1-y)^{-1} = \frac{y}{2(1-y)^2} - \frac{1}{2(1-y)} = (*)$$

$$\boxed{\begin{array}{l} u = y \longrightarrow u' = 1 \\ v' = (1-y)^{-3} \longrightarrow v = -\dfrac{(1-y)^{-2}}{-2} = \dfrac{1}{2}(1-y)^{-2} \end{array}}$$

$$(*) = \frac{y-(1-y)}{2(1-y)^2} = \frac{2y-1}{2(1-y)^2}$$

2 $\boxed{y' + f(x)\, y^2 + g(x)\, y = h(x)}$

$$y = y_1 + \frac{1}{z} \longrightarrow y' = y_1' - \frac{z'}{z^2}$$

$$y_1' - \frac{z'}{z^2} + f(x)\left(y_1^2 + 2 y_1 \frac{1}{z} + \frac{1}{z^2}\right) + g(x)\left(y_1 + \frac{1}{z}\right) = h(x)$$

Se $y_1(x)$ é solução da equação dada, então:

$$y_1' + f(x)\, y_1^2 + g(x)\, y_1 = h(x) \quad \text{Logo:}$$

$$\underbrace{y_1' + f(x)\, y_1^2 + g(x)\, y_1}_{\cancel{h(x)}} - \frac{z'}{z^2} + \frac{2 y_1}{z} f(x) + \frac{f(x)}{z^2} + \frac{g(x)}{z} \cancel{= h(x)}$$

$$\Leftrightarrow -\frac{z'}{z^2} + \frac{2 y_1 z f(x)}{z^2} + \frac{f(x)}{z^2} + \frac{g(x)\, z}{z^2} = 0$$

$$\Leftrightarrow -z' + 2 y_1 z f(x) + g(x)\, z = -f(x)$$

$$\Leftrightarrow z' + \left[-2 y_1 f(x) - g(x)\right] z = f(x)$$

É uma equação linear.

$$\boxed{3 \quad e^x \cdot \sec x - tg\, y + \frac{dy}{dx} = 0}$$

a) $\lambda = e^{-ax} \cdot \cos y$ é factor integrante

$$\left(e^x \cdot \sec x \cdot e^{-ax} \cos y - tg\, y \, e^{-ax} \cos y \right) dx + e^{-ax} \cos y \, dy = 0$$

$$\Leftrightarrow \left(e^{x-ax} - \operatorname{sen} y \, e^{-ax} \right) dx + e^{-ax} \cos y \, dy = 0$$

$$\frac{\partial P}{\partial y} = -\cos y \, e^{-ax} \qquad \frac{\partial Q}{\partial x} = -a \, e^{-ax} \cos y$$

Para que $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ então $\boxed{a = 1}$

b) $\boxed{\lambda = e^{-x} \cos y}$

$$\left(1 - \operatorname{sen} y \, e^{-x} \right) dx + e^{-x} \cos y \, dy = 0$$ é dif exacta

$$F_1 = \int \left(1 - \operatorname{sen} y \, e^{-x} \right) dx = x + \operatorname{sen} y \, e^{-x}$$

$$F_2 = \int e^{-x} \cos y \, dy = e^{-x} \operatorname{sen} y$$

$$\boxed{x + \operatorname{sen} y \, e^{-x} = C}$$

4 $\boxed{y''' - y'' - 5y' - 3y = 2e^{-x} + \operatorname{sen} x}$

$k^3 - k^2 - 5k - 3 = 0$

	1	-1	-5	-3
-1		-1	2	3
	1	-2	-3	0

$k^2 - 2k - 3 = 0$

$k = -1$ (duplo) $\vee$ $k = 3$ $\qquad \Leftrightarrow k = -1 \vee k = 3$

$$\boxed{y_H = (c_1 + c_2 x)e^{-x} + c_3 e^{3x}}$$

$Q_1(x) = 2e^{-x} \longrightarrow y_{p_1} = ax^2 e^{-x}$

$$y'_{p_1} = 2ax\, e^{-x} + ax^2(-e^{-x}) = e^{-x}(-ax^2 + 2ax)$$

$$y''_{p_1} = -e^{-x}(-ax^2 + 2ax) + e^{-x}(-2ax + 2a) = e^{-x}(ax^2 - 4ax + 2a)$$

$$y'''_{p_1} = -e^{-x}(ax^2 - 4ax + 2a) + e^{-x}(2ax - 4a) = e^{-x}(-ax^2 + 6ax - 6a)$$

$$e^{-x}(-\cancel{ax^2} + \cancel{6ax} - 6a) - e^{-x}(\cancel{ax^2} - \cancel{4ax} + 2a) - 5e^{-x}(-\cancel{ax^2} + \cancel{2ax}) - 3\cancel{ax^2}e^{-x} = 2e^{-x}$$

$$\Leftrightarrow -8a = 2 \Leftrightarrow a = -\frac{1}{4} \longrightarrow \boxed{y_{p_1} = \frac{-x^2 e^{-x}}{4}}$$

$Q_2(x) = \operatorname{sen} x \longrightarrow y_p = a \operatorname{sen} x + b \cos x$

$$y'_p = a\cos x - b \operatorname{sen} x$$

$$y''_p = -a \operatorname{sen} x - b\cos x$$

$$y'''_p = -a\cos x + b \operatorname{sen} x$$

$$-a\cos x + b\operatorname{sen} x + a\operatorname{sen} x + b\cos x - 5a\cos x + 5b\operatorname{sen} x - 3a\operatorname{sen} x - 3b\cos x = \operatorname{sen} x$$

$$\begin{cases} -2a + 6b = 1 \\ -6a - 2b = 0 \end{cases} \Leftrightarrow \begin{cases} -20a = 1 \\ b = -3a \end{cases} \Leftrightarrow \begin{cases} a = -\frac{1}{20} \\ b = \frac{3}{20} \end{cases}$$

$$y_{p_2} = \frac{-\operatorname{sen} x + 3\cos x}{20}$$

$$y_{GNH} = (c_1 + c_2 x)e^{-x} + c_3 e^{3x} - \frac{x^2 e^{-x}}{4} + \frac{3\cos x - \operatorname{sen} x}{20}$$

5 $\quad \boxed{x^2 y'' + \alpha x y' + \beta y = 0 \qquad \alpha, \beta \in \mathbb{R}}$

$y = x^2$ e $y = x^3$ são soluções, logo:

$y = x^2 \longrightarrow y' = 2x \longrightarrow y'' = 2$

$2x^2 + 2\alpha x^2 + \beta x^2 = 0 \longrightarrow \boxed{2 + 2\alpha + \beta = 0}$

$y = x^3 \longrightarrow y' = 3x^2 \longrightarrow y'' = 6x$

$6x^3 + 3x^3\alpha + \beta x^3 = 0 \longrightarrow \boxed{6 + 3\alpha + \beta = 0}$

$$\begin{cases} 2\alpha + \beta = -2 \\ 3\alpha + \beta = -6 \end{cases} \Leftrightarrow \begin{cases} \alpha = -4 \\ \beta = 6 \end{cases}$$

$$\boxed{⑥ \quad x^6 y'' + (3x^5 + 4x^3) y' + 4y = \frac{1}{x^4}}$$

$$x = \frac{1}{\sqrt{t}} \Leftrightarrow x = t^{-1/2} \longrightarrow \frac{dx}{dt} = -\frac{1}{2} t^{-3/2} \qquad \frac{dt}{dx} = -2t^{3/2}$$

$$y' = \frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{dy}{dt}\left(-2t^{3/2}\right)$$

$$y'' = \frac{d^2y}{dx^2} = \left[\frac{d^2y}{dt^2}\left(-2t^{3/2}\right) - 3t^{1/2}\frac{dy}{dt}\right]\cdot\left(-2t^{3/2}\right) =$$

$$= 4t^3\frac{d^2y}{dt^2} + 6t^2\frac{dy}{dt}$$

$$t^{-3}\left[4t^3\frac{d^2y}{dt^2} + 6t^2\frac{dy}{dt}\right] + \left(3t^{-5/2} + 4t^{-3/2}\right)\left(-2t^{3/2}\frac{dy}{dt}\right) + 4y = t^2$$

$$\Leftrightarrow 4\frac{d^2y}{dt^2} + \cancel{\frac{6}{t}\frac{dy}{dt}} - \cancel{\frac{6}{t}\frac{dy}{dt}} - 8\frac{dy}{dt} + 4y = t^2$$

$$\Leftrightarrow \frac{d^2y}{dt^2} - 2\frac{dy}{dt} + y = \frac{t^2}{4}$$

$$k^2 - 2k + 1 = 0 \Leftrightarrow k = 1 \text{ (duplo)} \longrightarrow y_H = (c_1 + c_2 t)e^t$$

$$Q(t) = \frac{t^2}{4} \longrightarrow y_p = at^2 + bt + c \Rightarrow y_p' = 2at + b \rightarrow y_p'' = 2a$$

$$2a - 4at - 2b + at^2 + bt + c = \frac{t^2}{4} \longrightarrow \begin{cases} a = \frac{1}{4} \\ -4a + b = 0 \\ 2a - 2b + c = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} a = \frac{1}{4} \\ b = 1 \\ c = \frac{3}{2} \end{cases} \qquad \boxed{y_p = \frac{t^2}{4} + t + \frac{3}{2}}$$

$$\boxed{\begin{aligned} y_{GNH} &= (c_1 + c_2 t)e^t + \frac{t^2}{4} + t + \frac{3}{2} = \\ &= \left(c_1 + c_2\frac{1}{x^2}\right)e^{\frac{1}{x^2}} + \frac{1}{4x^4} + \frac{1}{x^2} + \frac{3}{2} \end{aligned}}$$

$$\boxed{7) \quad y_{k+3} + y_{k+2} - 2y_k = k + 2^k}$$

$m^3 + m^2 - 2 = 0$

	1	1	0	-2
1		1	2	2
	1	2	2	0

$m = 1 \vee m^2 + 2m + 2 = 0$

$m = \dfrac{-2 \pm \sqrt{4 - 4(2)}}{2} = -1 \pm i$

$\rightarrow \rho = \sqrt{2}$

$\theta = \dfrac{3\pi}{4}$

$$y_H = c_1 + (\sqrt{2})^k \left[c_2 \operatorname{sen}\left(\frac{3\pi}{4}k\right) + c_3 \cos\left(\frac{3\pi}{4}k\right) \right]$$

$Q_1(k) = k \longrightarrow y_{p_1} = ak^2 + bk$

$y_{k+2} = a(k^2 + 4k + 4) + b(k+2)$

$y_{k+3} = a(k^2 + 6k + 9) + b(k+3)$

$$\cancel{ak^2} + 6ak + 9a + \cancel{bk} + 3b + \cancel{ak^2} + a4k + 4a + \cancel{bk} + 2b - \cancel{2ak^2} - \cancel{2bk} = k$$

$\Leftrightarrow 10ak + 13a + 5b = k \rightarrow \begin{cases} 10a = 1 \\ 13a + 5b = 0 \end{cases} \Leftrightarrow \begin{cases} a = \frac{1}{10} \\ b = -\frac{13}{50} \end{cases}$

$$\boxed{y_{p_1} = \frac{k^2}{10} - \frac{13}{50}k}$$

$Q_2(k) = 2^k \longrightarrow y_{p_2} = a \cdot 2^k \quad y_{k+1} = a \cdot 2^k \cdot 4 \quad y_{k+3} = a2^k \cdot 8$

$8a2^k + 4a \cdot 2^k - 2a2^k = 2^k \longrightarrow 10a = 1 \Leftrightarrow a = \frac{1}{10}$

$$\boxed{y_{p_2} = \frac{1}{10} \cdot 2^k}$$

$$\boxed{y_k = c_1 + (\sqrt{2})^k \left[c_2 \operatorname{sen}\left(\frac{3\pi}{4}k\right) + c_3 \cos\left(\frac{3\pi}{4}k\right) \right] + \frac{k^2}{10} - \frac{13}{50}k + \frac{2^k}{10}}$$