CATÓLICA LISBON
SCHOOL OF BUSINESS & ECONOMICS

UNIVERSIDADE CATÓLICA PORTUGUESA

Matemática III

3 de Janeiro de 2013 **2012/2013** **1ºSemestre**

Grupo I
(6.0 valores)

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1) Resolva as equações diferenciais:

a) $y' + y^2 - 2ysenx + sen^2x - \cos x = 4$ Sugestão: faça $y = senx + \frac{1}{z}$

b) $(y - xy^2 \ln x)dx + xdy = 0$

c) $y \cos x x' + 2y - senx = 0$

Grupo II
(4.0 valores)

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2) Considere a equação diferencial

$$f(\frac{y}{x})dx + g(\frac{y}{x})dy + kx^{\alpha}(xdy - ydx) = 0, \text{ com k e } \alpha \in \mathbb{R}.$$

Mostre que a mudança de variável $y = xz$ transforma a equação dada numa equação de Bernoulli.

3) Considere a equação diferencial

$$\left[f'(x)g(y)h(y) + \varphi'(x)h^2(y)\right]dx + \left[f(x)g'(y)h(y) - f(x)h'(y)g(y)\right]dy = 0.$$

Sabendo que a equação dada admite factor integrante $\lambda = \lambda(y)$, resolva a equação.

Grupo III
(6.0 valores)

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4) Resolva a equação diferencial $y^{IV} - 4y''' + 5y'' = -\frac{senx}{2} + x$

5) Considere a equação diferencial $y'' + f(x)y' + g(x)y = h(x)$.

Mostre que a substituição $y = y_p z$, em que y_p é solução particular da equação homogénea, reduz a ordem da equação diferencial dada.

6) A equação diferencial $(x+1)y'' + xy' - y = (x+1)^2$ admite como solução particular da equação homogénea $y_p = e^{-x}$. Em face disto, resolva a equação.

Grupo IV
(4.0 valores)

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7) Resolva a equação diferencial

$$4x^3 y'' + (6x^2 + 4x\sqrt{x})y' + 2y = \frac{1}{\sqrt{x e^{\frac{1}{\sqrt{x}}}}}$$

Sugestão: faça $\sqrt{x} = \frac{1}{t}$

8) Resolva a equação com diferenças

$$y_{k+3} + y_{k+2} - 2y_k = \frac{k}{(-1)^k} - 1$$

Grupo I

1

$$\boxed{c)\ y\cos x\, dx + (2y - \operatorname{sen} x)\, dy = 0}$$

$$\frac{\partial P}{\partial y} = \cos x \qquad \frac{\partial Q}{\partial x} = -\cos x$$

$$\frac{\partial P/\partial y - \partial Q/\partial x}{P} = \frac{2\cos x}{y\cos x} = \frac{2}{y}$$

$$\lambda = e^{-\int \frac{2}{y}dy} = e^{-2\ln|y|} = \frac{1}{y^2}$$

$$\frac{\cos x}{y}\, dx + \left(\frac{2}{y} - \frac{\operatorname{sen} x}{y^2}\right) dy = 0 \quad \underline{\text{Dif. exacta}}$$

$$F_1 = \int \frac{\cos x}{y}\, dx = \frac{\operatorname{sen} x}{y}$$

$$F_2 = \int \left(\frac{2}{y} - \frac{\operatorname{sen} x}{y^2}\right) dy = 2\ln|y| + \frac{\operatorname{sen} x}{y}$$

Solução : $\boxed{2\ln|y| + \frac{\operatorname{sen} x}{y} = C}$

$$\boxed{b)\ (y - x y^2 \ln x)\, dx + x\, dy = 0}$$

$$x y' + y = x y^2 \ln x$$

$$\Leftrightarrow \quad y' + \frac{1}{x} y = y^2 \ln x \qquad \underline{\text{Eq. de Bernoulli}}$$

$$y^{-2} y' + \frac{1}{x} y^{-1} = \ln x \qquad \boxed{z = y^{-1}} \longrightarrow z' = -1\, y^{-2} y'$$

$$-z' + \frac{1}{x} z = \ln x$$

$$z' - \frac{1}{x} z = -\ln x \qquad \underline{\text{Eq. linear}}$$

$$\lambda = e^{\int -\frac{1}{x}dx} = e^{-\ln|x|} = \frac{1}{x}$$

$$z \cdot \frac{1}{x} = \int -\ln x \cdot \frac{1}{x}\, dx$$

$$\frac{z}{x} = -\frac{\ln^2 x}{2} + C$$

$$\boxed{\frac{1}{xy} + \frac{\ln^2 x}{2} = C}$$

a) $y' + y^2 - 2y\,\text{sen}\,x + \text{sen}^2 x - \cos x = 4$

faça: $y = \text{sen}\,x + \frac{1}{z}$

$$y = \text{sen}\,x + \frac{1}{z} \longrightarrow y' = \cos x - \frac{z'}{z^2}$$

Substituindo vem:

$$\cancel{\cos x} - \frac{z'}{z^2} + \cancel{\text{sen}^2 x} + \cancel{2\,\text{sen}\,x \cdot \frac{1}{z}} + \frac{1}{z^2} - \cancel{2\,\text{sen}^2 x} - \cancel{\frac{2\,\text{sen}\,x}{z}} + \cancel{\text{sen}^2 x} - \cancel{\cos x} = 4$$

$$-\frac{z'}{z^2} + \frac{1}{z^2} = 4$$

$$-\frac{dz}{dx} + 1 = 4z^2 \Leftrightarrow -dz + (1-4z^2)\,dx = 0$$

$$\Leftrightarrow \frac{1}{4z^2-1}\,dz + dx = 0$$

$$\underbrace{\int \frac{1}{4z^2-1}\,dz}_{C.A} + \int dx = C$$

$$\Leftrightarrow \frac{1}{4}\ln\left|\frac{2z-1}{2z+1}\right| + x = C \qquad \text{Com } z = \frac{1}{y-\text{sen}\,x}$$

$$\Leftrightarrow \ln\left|\frac{\frac{2}{y-\text{sen}\,x} - 1}{\frac{2}{y-\text{sen}\,x} + 1}\right| + 4x = C \Leftrightarrow \boxed{\ln\left|\frac{2-y+\text{sen}\,x}{2+y-\text{sen}\,x}\right| + 4x = C}$$

C.A: $P\,\frac{1}{(2z-1)(2z+1)} = P\,\frac{\frac{1}{2}}{2z-1} + P\,\frac{-\frac{1}{2}}{2z+1} =$

$$= \frac{1}{4}\ln|2z-1| - \frac{1}{4}\ln|2z+1| =$$

$$= \frac{1}{4}\ln\left|\frac{2z-1}{2z+1}\right| + C$$

Grupo II

$$\boxed{(2)\quad f\left(\frac{y}{x}\right) dx + g\left(\frac{y}{x}\right) dy + k\, x^{\alpha} (x\, dy - y\, dx) = 0}$$

$$\boxed{y = x \cdot z} \longrightarrow dy = z\, dx + x\, dz$$

$$\left[f\left(\frac{y}{x}\right) - k\, x^{\alpha} y \right] dx + \left[g\left(\frac{y}{x}\right) + k\, x^{\alpha+1} \right] dy = 0$$

Substituindo vem :

$$\left[f(z) - k\, x^{\alpha+1} z \right] dx + \left[g(z) + k\, x^{\alpha+1} \right] (z\, dx + x\, dz) = 0$$

$$\Leftrightarrow \left[f(z) - \cancel{k\, x^{\alpha+1} z} + z \cdot g(z) + \cancel{k\, z\, x^{\alpha+1}} \right] dx + \left[x\, g(z) + k\, x^{\alpha+2} \right] dz = 0$$

$$\Leftrightarrow \left[f(z) + z\, g(z) \right] dx + x \left[g(z) + k\, x^{\alpha+1} \right] dz = 0$$

$$\left[f(z) + z\, g(z) \right] \frac{dx}{dz} + x\, g(z) = -k\, x^{\alpha+2}$$

$$\left[f(z) + z\, g(z) \right] x' + g(z) \cdot x = -k\, x^{\alpha+2}$$

Equação de Bernoulli em x

3 $$\left[f'(x)\, g(y)\, h(y) + \varphi'(x)\, h^2(y)\right] dx +$$

$$+ \left[f(x)\, g'(y)\, h(y) - h'(y)\, g(y) \cdot f(x)\right] dy = 0$$

$$\frac{\partial P}{\partial y} = f'(x)\left[g'(y)\, h(y) + g(y)\, h'(y)\right] + \varphi'(x)\; 2\, h(y)\, h'(y)$$

$$\frac{\partial Q}{\partial x} = f'(x)\left[g'(y)\, h(y) - h'(y)\, g(y)\right]$$

$$\frac{\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}}{P} = \frac{2 f'(x)\, g(y) \cdot h'(y) + 2\varphi'(x)\, h(y)\, h'(y)}{f'(x)\, g(y)\, h(y) + \varphi'(x)\, h^2(y)} =$$

$$= \frac{2\, h'(y)\left[\cancel{f'(x)\, g(y) + \varphi'(x)\, h(y)}\right]}{h(y)\left[\cancel{f'(x)\, g(y) + \varphi'(x)\, h(y)}\right]} =$$

$$= 2\, \frac{h'(y)}{h(y)}$$

$$\lambda(y) = e^{-\int 2 \frac{h'(y)}{h(y)}\, dy} = e^{-2 \ln|h(y)|} =$$

$$= \frac{1}{h^2(y)}$$

$$\boxed{\text{Factor integrante} : \quad \lambda(y) = \frac{1}{h^2(y)}}$$

$$\left[\frac{f'(x)\, g(y)}{h(y)} + \varphi'(x)\right] dx + f(x)\left[\frac{g'(y)}{h(y)} - \frac{h'(y)\, g(y)}{h^2(y)}\right] dy = 0 \quad \text{dif exacta}$$

$$F_1 = \int \left[\frac{f'(x)\, g(y)}{h(y)} + \varphi'(x)\right] dx = \frac{f(x) \cdot g(y)}{h(y)} + \varphi(x)$$

$$F_2 = \int f(x)\left[\frac{h(y)\, g'(y) - h'(y)\, g(y)}{h^2(y)}\right] dy = \frac{f(x) \cdot g(y)}{h(y)}$$

$$\boxed{\frac{f(x) \cdot g(y)}{h(y)} + \varphi(x) = C}$$

Grupo III

4) $y^{IV} - 4y''' + 5y'' = x - \frac{\operatorname{sen} x}{2}$

$k^4 - 4k^3 + 5k^2 = 0 \Leftrightarrow k^2(k^2 - 4k + 5) = 0$

$\Leftrightarrow k = 0$ (dupla) $\vee$ $k = \frac{4 \pm \sqrt{-4}}{2} = 2 \pm i$

$$\boxed{y_H = C_1 + C_2 x + e^{2x}(C_3 \cos x + C_4 \operatorname{sen} x)}$$

$y_{p_1} = (ax+b) \cdot x^2 = ax^3 + bx^2$

$y'_{p_1} = 3ax^2 + 2bx \longrightarrow y''_p = 6ax + 2b \longrightarrow y'''_p = 6a$

Substi. vem: $-24a + 30ax + 10b = x \rightarrow \begin{cases} 30a = 1 \\ -24a + 10b = 0 \end{cases}$

$\begin{cases} a = \frac{1}{30} \\ b = \frac{2}{25} \end{cases}$ $\quad y_{p_1} = \frac{x^3}{30} + \frac{2}{25}x^2$

$y_{p_2} = a \operatorname{sen} x + b \cos x \quad y' = a \cos x - b \operatorname{sen} x \quad y'' = -a \operatorname{sen} x - b \cos x$

$y''' = -a \cos x + b \operatorname{sen} x \quad y^{IV} = a \operatorname{sen} x + b \cos x$

Substi. vem:

$a \operatorname{sen} x + b \cos x + 4a \cos x - 4b \operatorname{sen} x - 5a \operatorname{sen} x - 5b \cos x = -\frac{\operatorname{sen} x}{2}$

$\begin{cases} a - 4b - 5a = -\frac{1}{2} \\ b + 4a - 5b = 0 \end{cases} \Leftrightarrow \begin{cases} a = \frac{1}{16} \\ b = a \end{cases}$

$y_{p_2} = \frac{\operatorname{sen} x + \cos x}{16}$

$$\boxed{y_{GNH} = C_1 + C_2 x + e^{2x}(C_3 \cos x + C_4 \operatorname{sen} x) + \frac{x^3}{30} + \frac{2x^2}{25} + \frac{\operatorname{sen} x + \cos x}{16}}$$

$$\boxed{⑤ \quad y'' + f(x)\, y' + g(x)\, y = h(x)}$$

$$y = y_p \cdot z$$

Se y_p é solução da homogénea, então:

$$y_p \longrightarrow y_p' \longrightarrow y_p''$$

Substituindo vem: $y_p'' + f(x)\, y_p' + g(x)\, y_p = 0$

$$y = y_p\, z \longrightarrow y' = y_p' \cdot z + y_p \cdot z'$$

$$y'' = y_p'' z + y_p' \cdot z' + y_p' \cdot z' + y_p \cdot z'' =$$

$$= y_p'' \cdot z + 2\, y_p' \cdot z' + y_p \cdot z''$$

Substituindo na equação vem:

$$y_p'' z + 2 y_p' z' + y_p z'' + f(x)\left(y_p' z + y_p z' \right) + g(x) \cdot y_p \cdot z = h(x)$$

$$\underbrace{\left(y_p'' + f(x)\, y_p' + g(x) \cdot y_p \right)}_{0} \cdot z + y_p z'' + f(x)\, y_p\, z' + 2 y_p' z' = h(x)$$

$$y_p z'' + f(x)\, y_p \cdot z' + 2 y_p' z' = h(x)$$

Fazendo $w = z' \longrightarrow w' = z''$

$$y_p w' + f(x)\, y_p\, w + 2 y_p' w = h(x)$$

$$\boxed{y_p w' + \left[f(x)\, y_p + 2 y_p' \right] w = h(x)}$$

É uma equação de ordem inferior

$$\boxed{6 \quad (x+1)y'' + x y' - y = (x+1)^2 \qquad y_p = e^{-x}}$$

$$y = e^{-x}\cdot z \qquad y' = -e^{-x}z + e^{-x}z' \qquad y'' = e^{-x}\cdot z - e^{-x}z' - e^{-x}z' + e^{-x}z'' =$$
$$= e^{-x}z - 2e^{-x}z' + e^{-x}\cdot z''$$

Substituindo vem:

$$(x+1)\left(e^{-x}z - 2e^{-x}z' + e^{-x}z''\right) + x\left(-e^{-x}z + e^{-x}z'\right) - e^{-x}z = (x+1)^2$$

$$(x+1)e^{-x}z'' - 2(x+1)e^{-x}z' + \cancel{(x+1)e^{-x}z} - \cancel{xe^{-x}z} + xe^{-x}z' - \cancel{e^{-x}z} = (x+1)^2$$

$$(x+1)z''e^{-x} - (x+2)e^{-x}z' = (x+1)^2$$

$$z'' - \frac{x+2}{x+1}z' = (x+1)e^{x} \qquad z' = w \longrightarrow z'' = w'$$

$$w' - \frac{x+2}{x+1}w = (x+1)e^{x} \quad \text{Eq. linear}$$

$$\lambda = e^{-\int\left(1+\frac{1}{x+1}\right)dx} = e^{-x-\ln|x+1|} = e^{-x}\cdot\frac{1}{x+1}$$

$$\frac{w\cdot e^{-x}}{x+1} = \int \cancel{(x+1)}e^{x}\cdot e^{-x}\cdot\frac{1}{\cancel{x+1}}\,dx \iff \frac{we^{-x}}{x+1} = x + C$$

$$\iff w = (x+C)(x+1)e^{x} \iff w = \left(x^2 + x + C_1x + C_1\right)e^{x}$$

$$z' = \left(x^2+x+C_1x+C_1\right)e^{x} \quad \text{logo} \quad z = \int\left(x^2+x+C_1x+C_1\right)e^{x}\,dx$$

$$z = e^{x}\left(x^2 - x + C_1x + 1\right) + C_2$$

$$\frac{y}{e^{-x}} = e^{x}\left(x^2 - x + C_1x + 1\right) + C_2 \iff \boxed{y = x^2 - x + C_1x + 1 + C_2e^{-x}}$$

CA:

$$P\left(x^2+x+C_1x+C_1\right)e^{x} = e^{x}\left(x^2+x+C_1x+C_1\right) - P\,e^{x}\left(2x+1+C_1\right) =$$

$u = x^2+x+C_1x+C_1 \longrightarrow u' = 2x+1+C_1$	
$v' = e^{x} \longrightarrow v = e^{x}$	
$u = 2x+1+C_1 \longrightarrow u' = 2$	
$v' = e^{x} \longrightarrow v = e^{x}$	

$$= e^{x}\left(x^2+x+C_1x+C_1\right) - \left[e^{x}\left(2x+1+C_1\right) - P\,2\,e^{x}\right] =$$
$$= e^{x}\left(x^2+x+C_1x+C_1\right) - e^{x}\left(2x+1+C_1\right) + 2e^{x} + C_2 =$$
$$= e^{x}\left(x^2 - x + C_1x + 1\right) + C_2$$

Grupo IV

8

7) $$4x^3 y'' + \left(6x^2 + 4x\sqrt{x}\right) y' + 2y = \frac{1}{\sqrt{x}\, e^{\frac{1}{\sqrt{x}}}}$$

faça: $\sqrt{x} = \dfrac{1}{t}$

$$x = \frac{1}{t^2} \qquad \frac{dx}{dt} = -\frac{2}{t^3} \qquad \frac{dt}{dx} = -\frac{t^3}{2}$$

$$y' = \frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx} = \boxed{\frac{dy}{dt}\cdot\left(-\frac{t^3}{2}\right)} \qquad A - t - x$$

$$y'' = \frac{d^2y}{dx^2} = \frac{dA}{dt}\cdot\frac{dt}{dx} = \left[\frac{d^2y}{dt^2}\left(-\frac{t^3}{2}\right) - \frac{3t^2}{2}\cdot\frac{dy}{dt}\right]\cdot\left(-\frac{t^3}{2}\right) =$$

$$= \frac{t^6}{4}\frac{d^2y}{dt^2} + \frac{3t^5}{4}\frac{dy}{dt}$$

Substituindo vem:

$$\frac{4}{t^6}\left(\frac{t^6}{4}\frac{d^2y}{dt^2} + 3\frac{t^5}{4}\frac{dy}{dt}\right) + \left(6\frac{1}{t^4} + 4\cdot\frac{1}{t^3}\right)\left(\frac{-t^3}{2}\frac{dy}{dt}\right) + 2y = \frac{t}{e^t}$$

$$\Leftrightarrow \frac{d^2y}{dt^2} + \cancel{\frac{3}{t}\frac{dy}{dt}} - \cancel{\frac{3}{t}\frac{dy}{dt}} - 2\frac{dy}{dt} + 2y = t\,e^{-t}$$

$$\Leftrightarrow \frac{d^2y}{dt^2} - 2\frac{dy}{dt} + 2y = t\,e^{-t}$$

$$k^2 - 2k + 2 = 0 \Leftrightarrow k = \frac{2 \pm \sqrt{4-8}}{2} = 1 \pm i$$

$$\boxed{y_H = e^t\left(c_1 \cos t + c_2 \operatorname{sen} t\right)}$$

$$Q(t) = te^{-t} \longrightarrow y_p = (at+b)e^{-t}$$

$$y_p' = a e^{-t} + (at+b)(-e^{-t}) =$$

$$= e^{-t}(-at - b + a)$$

$$y_p'' = -e^{-t}(-at-b+a) + e^{-t}(-a) =$$

$$= e^{-t}(at + b - 2a)$$

$$at + b - 2a + 2at + 2b - 2a + 2at + 2b = t$$

$$5at - 4a + 5b = t \qquad \begin{cases} 5a = 1 \\ -4a + 5b = 0 \end{cases} \Leftrightarrow \begin{cases} a = 1/5 \\ b = 4/25 \end{cases} \qquad \boxed{y_p = \left(\frac{1}{5}t + \frac{4}{25}\right)e^{-t}}$$

$$y_{GNH} = e^{\frac{1}{\sqrt{x}}}\left(c_1 \cos\frac{1}{\sqrt{x}} + c_2 \operatorname{sen}\frac{1}{\sqrt{x}}\right) + \left(\frac{1}{5}\cdot\frac{1}{\sqrt{x}} + \frac{4}{25}\right)e^{-\frac{1}{\sqrt{x}}}$$

8 $y_{k+3} + y_{k+2} - 2y_k = \frac{k}{(-1)^k} - 1$

$$\boxed{y_{k+3} + y_{k+2} - 2y_k = (-1)^k \cdot k - 1}$$

$m^3 + m^2 - 2 = 0$

$$\begin{array}{c|cccc} & 1 & 1 & 0 & -2 \\ 1 & & 1 & 2 & 2 \\ \hline & 1 & 2 & 2 & \underline{|0} \end{array}$$

$(m-1)(m^2+2m+2) = 0$

$m = 1 \lor m = \frac{-2 \pm \sqrt{4-8}}{2} =$

$$\boxed{y_H = c_1 + \sqrt{2}^k \left(c_2 \cos \frac{3k\pi}{4} + c_3 \operatorname{sen} \frac{3k\pi}{4} \right)}$$

$m = -1 \pm i$

$\rho = \sqrt{1+1} = \sqrt{2}$

$\operatorname{tg}\theta = -1 \quad \theta = \frac{3\pi}{4}$

$Q_1(k) = -1 \longrightarrow y_k = ak$

$y_{k+2} = a(k+2)$

$y_{k+3} = a(k+3)$

$\cancel{ak} + 3a + \cancel{ak} + 2a - 2\cancel{ak} = -1 \iff 5a = -1$

$\iff a = -\frac{1}{5} \quad \boxed{y_k = -\frac{1}{5}k}$

$Q_2(k) = (-1)^k k \longrightarrow y_k = (ak+b)(-1)^k$

$y_{k+2} = (ak+2a+b)(-1)^k(-1)^2 = (ak+2a+b)(-1)^k$

$y_{k+3} = (ak+3a+b)(-1)^k(-1) = (-ak-3a-b)(-1)^k$

$-\cancel{ak} - 3a - \cancel{b} + \cancel{ak} + 2a + \cancel{b} - 2ak - 2b = k$

$$\begin{cases} -2a = 1 \\ -a - 2b = 0 \end{cases} \iff \begin{cases} a = -\frac{1}{2} \\ b = \frac{1}{4} \end{cases} \quad \boxed{y_k = \left(-\frac{k}{2} + \frac{1}{4}\right)(-1)^k}$$

$$\boxed{y_k = c_1 + \sqrt{2}^k \left[c_2 \cos\left(\frac{3k\pi}{4}\right) + c_3 \operatorname{sen}\left(\frac{3k\pi}{4}\right) \right] + \left(\frac{-k}{2} + \frac{1}{4}\right)(-1)^k - \frac{1}{5}k}$$