CATÓLICA LISBON
BUSINESS & ECONOMICS

Matemática III

2012-2013 • 1º Semestre • 1ª Frequência
30 de Outubro de 2012

O teste tem a duração de 2h30mn. Deve resolver os grupos em folhas separadas.

Grupo I (9.5 valores)

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1. Calcule as seguintes primitivas

a) $P\,\dfrac{\operatorname{arcsen} x}{\sqrt{x+1}}$ b) $P\,\dfrac{\ln(x^2-1)}{(x-1)^3}$ c) $P\,\dfrac{\operatorname{sen}^5(3x)}{\sqrt{2\cos(3x)}}$

d) $P\,\dfrac{\operatorname{arctg}\left(\sqrt[3]{x^2}\right)}{\sqrt[3]{x}+x\sqrt[3]{x^2}}$ e) $P\,\dfrac{\operatorname{sen}^2 x}{\operatorname{sen} x\cos x+2\cos^2 x}$ (***Sugestão*****: fazer** $t=\operatorname{tg} x$)

Grupo II (5.5 valores)

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2. Mostre que $P\operatorname{sen}^n x = \dfrac{n-1}{n} P\operatorname{sen}^{n-2} x - \dfrac{\cos x \operatorname{sen}^{n-1} x}{n}$ para $n>2$.

3. Mostre que $\displaystyle\int_{-a}^{b-a} \frac{e^{-\pi x^2}}{x^2}\,dx + \int_{-a+b}^{a} \frac{e^{-\pi x^2}}{x^2}\,dx = 2\int_{0}^{a} \frac{e^{-\pi x^2}}{x^2}\,dx$ com $a,b\in\mathbb{R}^+$.

4. Seja $\psi(x) = \displaystyle\int_{\frac{5x}{2}}^{\frac{x+x^2}{2}} f(t)\sqrt{16-3x}\,dt$ onde $f(x)$ é uma função contínua tal que $f(10)=5$.

Determine o valor de $\alpha\in\mathbb{R}$ de modo que: $\alpha\,\psi'(4)-(\alpha-1)^3\,\psi(4)=1$.

Grupo III (5.0 valores)

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5. Considere $I=\displaystyle\int_{-2}^{-1} dx \int_{-\sqrt{-x^2-2x}}^{x+2} f(x,y)\,dy + \int_{-1}^{0} dx \int_{-\sqrt{-x^2-2x}}^{x^2} f(x,y)\,dy$.

a) Inverta a ordem de integração.

b) Calcule o valor de I com $f(x,y)=x^2+y^2$.

6. Calcule o integral $\displaystyle\iint_D \frac{y^3}{x}\,dx\,dy$ onde $D=\begin{cases} xy\ge 1\\ xy\le 3\\ x\le 2y\\ y\le 2x\end{cases}$

Resolução

Grupo I

1.

a) $P\,\dfrac{\operatorname{arcsen} x}{\sqrt{x+1}}$ partes:

$$u = \operatorname{arcsen} x \qquad u' = \frac{1}{\sqrt{1-x^2}}$$

$$v' = (x+1)^{-\frac{1}{2}} \qquad v = 2\sqrt{x+1}$$

$$P\,\frac{\operatorname{arcsen} x}{\sqrt{x+1}} = 2\sqrt{x+1}\operatorname{arcsen} x - 2P\frac{\sqrt{x+1}}{\sqrt{(1-x)(1+x)}} = 2\sqrt{x+1}\operatorname{arcsen} x - 2P(1-x)^{-\frac{1}{2}} =$$

$$= 2\sqrt{x+1}\operatorname{arcsen} x + 4\sqrt{1-x} + C$$

b) $P\dfrac{\ln(x^2-1)}{(x-1)^3}$ partes:

$$u = \ln(x^2-1) \qquad u' = \frac{2x}{x^2-1}$$

$$v' = (x-1)^{-3} \qquad v = \frac{-1}{2(x-1)^2}$$

$$P\frac{\ln(x^2-1)}{(x-1)^3} = -\frac{\ln(x^2-1)}{2(x-1)^2} + P\frac{x}{(x-1)^3(x+1)} = -\frac{\ln(x^2-1)}{2(x-1)^2} + P\left[\frac{A}{(x-1)^3} + \frac{B}{(x-1)^2} + \frac{C}{x-1} + \frac{D}{x+1}\right]$$

Com $A = \left[\dfrac{x}{x+1}\right]_{x=1} = \dfrac{1}{2}$, $\quad D = \left[\dfrac{x}{(x-1)^3}\right]_{x=-1} = \dfrac{1}{8}$ fica

$$x = \frac{1}{2}(x+1) + B(x^2-1) + C(x-1)^2(x+1) + \frac{1}{8}(x-1)^3$$

Coeficientes de x^0: $\quad 0 = \dfrac{1}{2} - B + C - \dfrac{1}{8}$

Coeficientes de x^3: $\quad 0 = C + \dfrac{1}{8}$ $\qquad$ logo $\quad B = \dfrac{1}{4},\quad C = -\dfrac{1}{8}$

Então $$P\frac{\ln(x^2-1)}{(x-1)^3} = -\frac{\ln(x^2-1)}{2(x-1)^2} + \frac{1}{2}P(x-1)^{-3} + \frac{1}{4}P(x-1)^{-2} - \frac{1}{8}P\frac{1}{x-1} + \frac{1}{8}P\frac{1}{x+1} =$$

$$= -\frac{\ln(x^2-1)}{2(x-1)^2} - \frac{1}{4(x-1)^2} - \frac{1}{4(x-1)} + \frac{1}{8}\ln\left|\frac{x+1}{x-1}\right| + C$$

c) $P\,\dfrac{\operatorname{sen}^5(3x)}{\sqrt{2\cos(3x)}}$

Hipótese 1 (quase imediata)

$$P\,\frac{\operatorname{sen}^5(3x)}{\sqrt{2\cos(3x)}} = \frac{1}{\sqrt{2}}P\frac{\operatorname{sen} 3x(1-\cos^2 3x)^2}{\sqrt{\cos(3x)}} = \frac{1}{\sqrt{2}}P\left[\operatorname{sen} 3x\cos^{-\frac{1}{2}} 3x - 2\operatorname{sen} 3x\cos^{\frac{3}{2}} 3x + \operatorname{sen} 3x\cos^{\frac{7}{2}}\right] =$$

$$= \frac{1}{\sqrt{2}}\left[-\frac{2}{3}\sqrt{\cos 3x} + \frac{4}{15}\sqrt{\cos^5 3x} - \frac{2}{27}\sqrt{\cos^9 3x}\right] + C$$

Hipótese 2 (substituição)

$P\ \frac{\operatorname{sen}^5(3x)}{\sqrt{2\cos(3x)}}$ substituição: $t=\sqrt{\cos 3x}\ ,\quad t^2=\cos 3x,\quad \sin 3x=\sqrt{1-t^4}$

$$x=\frac{1}{3}\arccos t^2,\qquad x'=-\frac{1}{3}\,\frac{2t}{\sqrt{1-t^4}}$$

Então:

$$P\ \frac{\operatorname{sen}^5(3x)}{\sqrt{2\cos(3x)}}=\frac{1}{\sqrt{2}}P\frac{\sqrt{1-t^4}^{\,5}}{t}\,\frac{-2t}{3\sqrt{1-t^4}}=-\frac{\sqrt{2}}{3}\,P(1-t^4)^2=-\frac{\sqrt{2}}{3}\,P\left(1-2t^4+t^8\right)=$$
$$=-\frac{\sqrt{2}}{3}\left(\sqrt{\cos 3x}-\frac{2}{5}\sqrt{\cos 3x}^{\,5}+\frac{1}{9}\sqrt{\cos 3x}^{\,9}\right)+C$$

d) $P\ \frac{\operatorname{arctg}\left(\sqrt[3]{x^2}\right)}{\sqrt[3]{x}+x\sqrt[3]{x^2}}$

Hipótese 1 (Imediata)

$P\ \frac{\operatorname{arctg}\left(\sqrt[3]{x^2}\right)}{\sqrt[3]{x}+x\sqrt[3]{x^2}}=P\ \frac{\operatorname{arctg} x^{\frac{2}{3}}}{x^{\frac{1}{3}}+x^{\frac{5}{3}}}$ e, como $\left[\operatorname{arctg}\left(\sqrt[3]{x^2}\right)\right]'=\frac{\frac{2}{3}x^{-\frac{1}{3}}}{1+x^{\frac{4}{3}}}=\frac{2}{3}\,\frac{1}{x^{\frac{1}{3}}+x^{\frac{5}{3}}}$

é uma potência de $\operatorname{arctg}\left(\sqrt[3]{x^2}\right)$:

$$P\ \frac{\operatorname{arctg}\left(\sqrt[3]{x^2}\right)}{\sqrt[3]{x}+x\sqrt[3]{x^2}}=\frac{3}{2}P\ \underbrace{\frac{2}{3}\,\frac{1}{x^{\frac{1}{3}}+x^{\frac{5}{3}}}}_{u'}\ \underbrace{\operatorname{arctg}\left(\sqrt[3]{x^2}\right)}_{u}=\frac{3}{2}\,\frac{\operatorname{arctg}^2\left(\sqrt[3]{x^2}\right)}{2}+C=\frac{3}{4}\operatorname{arctg}^2\left(\sqrt[3]{x^2}\right)+C$$

Hipótese 2 (substituição)

$P\ \frac{\operatorname{arctg}\left(\sqrt[3]{x^2}\right)}{\sqrt[3]{x}+x\sqrt[3]{x^2}}$ fazendo $t=\sqrt[3]{x^2}=x^{\frac{2}{3}},\quad x=\sqrt{t^3}\ ,\quad x'=\frac{3}{2}\sqrt{t}$ logo:

$$P\ \frac{\operatorname{arctg}\left(\sqrt[3]{x^2}\right)}{\sqrt[3]{x}+x\sqrt[3]{x^2}}=P\frac{\operatorname{arctg} t}{(t^{3/2})^{1/3}+t\,t^{3/2}}\,\frac{3}{2}\,t^{\frac{1}{2}}=\frac{3}{2}P\frac{\operatorname{arctg} t}{1+t^2}=\frac{3}{2}\,\frac{\operatorname{arctg}^2 t}{2}+C=\frac{3}{4}\operatorname{arctg}^2\left(\sqrt[3]{x^2}\right)+C$$

e) $P\ \frac{\operatorname{sen}^2 x}{\operatorname{sen} x\cos x+2\cos^2 x}$ seja $t=\operatorname{tg} x,\quad x=\operatorname{arctg} t,\quad x'=\frac{1}{1+t^2}$

por outro lado:

$$\operatorname{tg}^2 x+1=\sec^2 x\ \Leftrightarrow\ \cos^2 x=\frac{1}{\operatorname{tg}^2 x+1}\quad \text{e}\quad \cos x=\frac{1}{\sqrt{t^2+1}},\quad \operatorname{sen} x=\sqrt{1-\cos^2 x}=\frac{t}{\sqrt{t^2+1}}$$

portanto

$$P\,\frac{\operatorname{sen}^2 x}{\operatorname{sen} x\cos x+2\cos^2 x}=P\,\frac{\dfrac{t^2}{t^2+1}}{\dfrac{t}{\sqrt{t^2+1}}\dfrac{1}{\sqrt{t^2+1}}+2\dfrac{1}{t^2+1}}\,\frac{1}{t^2+1}=P\frac{t^2}{(t^2+1)(t+2)}=P\left[\frac{A}{t+2}+\frac{Bt+C}{t^2+1}\right]$$

com $A=\left[\dfrac{t^2}{t^2+1}\right]_{t=-2}=\dfrac{4}{5}$ fica $t^2=\dfrac{4}{5}(t^2+1)+(Bt+C)(t+2)\ \Rightarrow\ \begin{cases}1=\dfrac{4}{5}+B\\[4pt]0=\dfrac{4}{5}+2C\end{cases}$ $B=\dfrac{1}{5},\ C=-\dfrac{2}{5}$

logo

$$P\,\frac{\operatorname{sen}^2 x}{\operatorname{sen} x\cos x+2\cos^2 x}=\frac{4}{5}P\frac{1}{t+2}+\frac{1}{5}P\frac{t-2}{t^2+1}=\frac{4}{5}\ln|t+2|+\frac{1}{10}P\frac{2t-4}{t^2+1}=$$

$$=\frac{4}{5}\ln|t+2|+\frac{1}{10}P\frac{2t}{t^2+1}-\frac{2}{5}P\frac{1}{t^2+1}=\frac{4}{5}\ln|\operatorname{tg}x+2|+\frac{1}{10}\ln\left|\operatorname{tg}^2 x+1\right|-\frac{2}{5}x+C=$$

$$=\frac{4}{5}\ln|\operatorname{tg}x+2|+\frac{1}{10}\ln\left|\sec^2 x\right|-\frac{2}{5}x+C$$

Grupo II

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2. $P\operatorname{sen}^n x = P\operatorname{sen} x\operatorname{sen}^{n-1} x$ partes: $u=\operatorname{sen}^{n-1}x$ $\quad u'=(n-1)\operatorname{sen}^{n-2}x\cos x$; $v'=\operatorname{sen} x$ $\quad v=-\cos x$

$$P\operatorname{sen}^n x=-\cos x\operatorname{sen}^{n-1}x+(n-1)P\operatorname{sen}^{n-2}x\cos^2 x=-\cos x\operatorname{sen}^{n-1}x+(n-1)P\operatorname{sen}^{n-2}x(1-\operatorname{sen}^2 x)$$

$$P\operatorname{sen}^n x=-\cos x\operatorname{sen}^{n-1}x+(n-1)P\operatorname{sen}^{n-2}x-(n-1)P\operatorname{sen}^n x$$

$$n\,P\operatorname{sen}^n x=(n-1)P\operatorname{sen}^{n-2}x-\cos x\operatorname{sen}^{n-1}x$$

logo:

$$P\operatorname{sen}^n x=\frac{n-1}{n}P\operatorname{sen}^{n-2}x-\frac{\cos x\operatorname{sen}^{n-1}x}{n}\quad\text{c.q.d.}$$

3. $\displaystyle\int_{-a}^{b-a}\frac{e^{-\pi x^2}}{x^2}\,dx+\int_{-a+b}^{a}\frac{e^{-\pi x^2}}{x^2}\,dx$ tem-se $-a\le b-a\le a$ logo

$$\int_{-a}^{b-a}\frac{e^{-\pi x^2}}{x^2}\,dx+\int_{-a+b}^{a}\frac{e^{-\pi x^2}}{x^2}\,dx=\int_{-a}^{a}\frac{e^{-\pi x^2}}{x^2}\,dx=\int_{-a}^{0}\frac{e^{-\pi x^2}}{x^2}\,dx+\int_{0}^{a}\frac{e^{-\pi x^2}}{x^2}\,dx$$

fazendo no primeiro integral $x=-t$ fica:

$$-\int_{a}^{0}\frac{e^{-\pi t^2}}{t^2}\,dt+\int_{0}^{a}\frac{e^{-\pi x^2}}{x^2}\,dx$$

e sendo *t* variável muda:

$$-\int_{a}^{0}\frac{e^{-\pi t^2}}{t^2}\,dt+\int_{0}^{a}\frac{e^{-\pi x^2}}{x^2}\,dx=-\int_{a}^{0}\frac{e^{-\pi x^2}}{x^2}\,dt+\int_{0}^{a}\frac{e^{-\pi x^2}}{x^2}\,dx=\int_{0}^{a}\frac{e^{-\pi x^2}}{x^2}\,dt+\int_{0}^{a}\frac{e^{-\pi x^2}}{x^2}\,dx=$$

$$=2\int_{0}^{a}\frac{e^{-\pi x^2}}{x^2}\,dx\quad\text{c.q.d.}$$

4. $\psi(x) = \displaystyle\int_{\frac{5x}{2}}^{\frac{x+x^2}{2}} f(t)\sqrt{16-3x}\ dt$

$$\psi'(x) = \frac{-3}{2\sqrt{16-3x}} \int_{\frac{5x}{2}}^{\frac{x+x^2}{2}} f(t)\ dt + \sqrt{16-3x}\left[f\left(\frac{x+x^2}{2}\right)\left(x+\frac{1}{2}\right) - f\left(\frac{5x}{2}\right)\frac{5}{2}\right] =$$

$$= \frac{-3}{2\sqrt{16-3x}} \int_{\frac{5x}{2}}^{\frac{x+x^2}{2}} f(t)\ dt + f\left(\frac{x+x^2}{2}\right)\left(x+\frac{1}{2}\right)\sqrt{16-3x} - \frac{5}{2} f\left(\frac{5x}{2}\right)\sqrt{16-3x}$$

então,

$$\psi'(4) = -\frac{3}{4}\int_{10}^{10} f(t)dt + 9f(10) - 5f(10) = 4f(10) = 20 \quad \text{e} \quad \psi(4) = \int_{10}^{10} f(t)dt = 0$$

portanto deve ter-se

$$20\,\alpha - (\alpha-1)^3 \times 0 = 1 \ \Rightarrow \ \alpha = \frac{1}{20}$$

Grupo III

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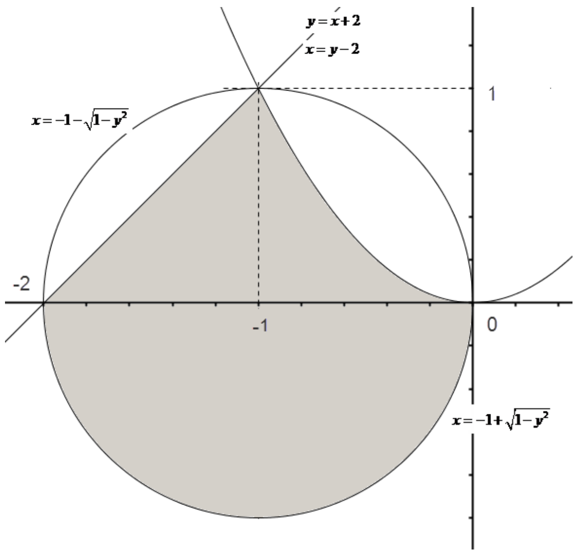
5. $I = \displaystyle\int_{-2}^{-1} dx \int_{-\sqrt{-x^2-2x}}^{x+2} f(x,y)\ dy + \int_{-1}^{0} dx \int_{-\sqrt{-x^2-2x}}^{x^2} f(x,y)\ dy$

a) $y = -\sqrt{-x^2-2x} \ \rightarrow \ (x-1)^2 + y^2 = 1 \quad \text{e} \quad x = -1 \pm \sqrt{1-y^2}$



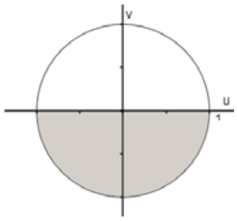
Pela ordem inversa fica

$$I = \int_{-1}^{0} dy \int_{1-\sqrt{1-y^2}}^{1+\sqrt{1-y^2}} f(x,y)\ dx + \int_{0}^{1} dy \int_{y-2}^{-\sqrt{y}} f(x,y)\ dx$$

b) Se $f(x,y) = x^2 + y^2$ fica

$$I = \underbrace{\int_{-1}^{0} dy \int_{1-\sqrt{1-y^2}}^{1+\sqrt{1-y^2}} (x^2+y^2)\, dx}_{\text{Centra-se e calcula-se em coordenadas polares}} + \underbrace{\int_{0}^{1} dy \int_{y-2}^{-\sqrt{y}} (x^2+y^2)\, dx}_{\text{Calcula-se em coordenadas cartesianas}} = I_1 + I_2$$

I_1: fazendo $\begin{cases} x+1=u \\ y=v \end{cases}$, $\dfrac{\partial(x,y)}{\partial(u,v)} = 1$ o domínio fica $\begin{cases} u^2+v^2 \le 1 \\ v \le 0 \end{cases}$



e passando a polares:

$$I_1 = \int_0^1 dr \int_{-\pi}^{0} r(r^2cos^2\theta - 2r\cos\theta + r^2 sin^2\theta + 1)\, d\theta =$$

$$= \int_0^1 r^3 dr \int_{-\pi}^{0} d\theta \;\; \cancel{- 2\int_0^1 r^2 dr \underbrace{\int_{-\pi}^{0} \cos\theta d\theta}_{=0}} \;\; + \int_0^1 r\, dr \int_{-\pi}^{0} d\theta = \frac{\pi}{4} + \frac{\pi}{2} = \frac{3\pi}{4}$$

I_2: $$\int_0^1 dy \int_{y-2}^{-\sqrt{y}} (x^2+y^2)\, dx = \int_0^1 \left[\frac{x^3}{3} + xy^2\right]_{y-2}^{-\sqrt{y}} dy = \int_0^1 \left[-\frac{1}{3} y^{\frac{3}{2}} - y^{\frac{5}{2}} - \frac{1}{3}(y-2)^3 - y^2(y-2)\right] dy =$$

$$= -\frac{2}{15}\left(y^{\frac{5}{2}}\right)_0^1 - \frac{2}{7}\left(y^{\frac{7}{2}}\right)_0^1 - \frac{1}{12}\left[(y-2)^4\right]_0^1 - \frac{1}{4}\left(y^4\right)_0^1 + \frac{2}{3}\left(y^3\right)_0^1 = -\frac{2}{15} - \frac{2}{7} + \frac{15}{12} - \frac{1}{4} + \frac{2}{3} = \frac{131}{105}$$

portanto:

$$I = \frac{3\pi}{4} + \frac{131}{105}$$

6. $\displaystyle\iint_D \frac{y^3}{x}\, dx\, dy$ com $D = \begin{cases} xy \ge 1 \\ xy \le 3 \\ x \le 2y \\ y \le 2x \end{cases}$ ou seja $D = \begin{cases} 1 \le xy \le 3 \\ \frac{1}{2} \le \frac{y}{x} \le 2 \end{cases}$ (1º Q)

Se $\begin{cases} u = xy \\ v = \dfrac{y}{x} \end{cases}$, $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} y & x \\ -y/x^2 & 1/x \end{vmatrix}^{-1} = \dfrac{x}{2y} = \dfrac{1}{2v}$ e o integral fica:

$$\iint_{D'} \frac{y^2}{2}\, dudv = \frac{1}{2}\iint_{D'} \frac{xy^2}{x}\, dudv = \frac{1}{2}\iint_{D'} uv\; dudv \quad \text{com} \quad D' = \begin{cases} 1 \le u \le 3 \\ \frac{1}{2} \le v \le 2 \end{cases}$$

$$\iint_D \frac{y^3}{x}\, dx\, dy = \frac{1}{2}\iint_{D'} uv\; dudv = \frac{1}{2}\int_1^3 udu \int_{1/2}^{2} vdv = \frac{1}{8}\left(u^2\right)_1^3 \left(v^2\right)_{1/2}^{2} = \frac{15}{4}$$